Analytic formulas for the mechanical properties of radio waves emitted by an antenna array

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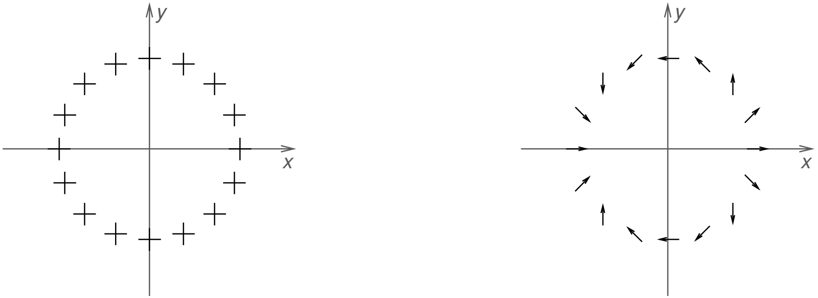

$\mathbf{I}$ $\mathbf{A}$ $\mathbf{B}$ $\mathbf{E}$ w $\mathbf{p}$ $\mathbf{j}$ $\mathbf{l}$ $\mathbf{s}$

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Vector potential

circular antenna array $n = 1 \ldots N$

$$\mathbf{r}_n = \begin{pmatrix} R\cos(2\pi\frac{n}{N}) \\ R\sin(2\pi\frac{n}{N}) \\ 0 \end{pmatrix} \qquad \mathbf{I}_n = \frac{1}{\sqrt{2}}\begin{pmatrix} I \\ ihI \\ 0 \end{pmatrix} e^{2\pi i l \frac{n}{N}} \qquad \mathbf{I}(t) = \mathbf{I}e^{-i\omega t}$$

vector potential

$$\mathbf{A}(\mathbf{r},t) = \sum_{n=1}^{N} \mathbf{A}_n(\mathbf{r}-\mathbf{r}_n,t) \qquad \mathbf{A}_n(\mathbf{r},t) = -\frac{\mu_0 d}{8\pi}\mathbf{I}_n(t)\frac{e^{ikr}}{r}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ ih \\ 0 \end{pmatrix} \mathcal{A}(r)\mathcal{C}(\theta,\phi)$$

$$\mathcal{A} = -\frac{I\mu_0 d}{8\pi}\frac{e^{ikr}}{r}$$

$$\mathcal{C} = \sum_{n=1}^{N} e^{2\pi i l \frac{n}{N}} e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}_n}$$

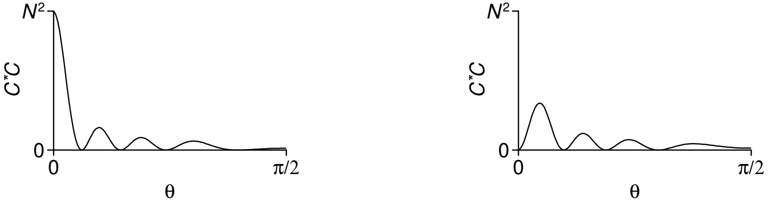
Radial and angular dependencies

$$\mathcal{A}(r) \propto \frac{1}{r}$$

$$\mathcal{C}(\theta,\phi) \simeq \frac{N}{2\pi}\int_0^{2\pi} d\Phi e^{il\Phi} e^{-ikR\sin\theta\cos(\Phi-\phi)}$$

$$\frac{\partial}{\partial\phi}\ln\mathcal{C} = il \qquad \frac{\partial}{\partial\theta}\ln\mathcal{C} = \text{real} \qquad \frac{\partial}{\partial\phi}(\mathcal{C}^*\mathcal{C}) = 0$$

$\mathcal{C}^*\mathcal{C}$



Fields

magnetic field

$$\mathbf{B} = \nabla \times \mathbf{A}$$

power series in r

$$\mathbf{B} = \mathbf{B}_{\text{far}} + \mathbf{B}_{\text{near}}$$

with

$$\mathbf{B}_{\text{far}} = ik\hat{\mathbf{r}} \times \mathbf{A}$$

$$\mathbf{B}_{\text{near}} = e^{ikr}\nabla \times (\mathbf{A}e^{-ikr})$$

electric field

$$\mathbf{E} = \frac{ic}{k}\nabla \times \mathbf{B}$$

$$\mathbf{E} = \mathbf{E}_{\text{far}} + \mathbf{E}_{\text{inter}} + \mathbf{E}_{\text{near}}$$

$$\mathbf{E}_{\text{far}} = ikc[\mathbf{A} - (\hat{\mathbf{r}} \cdot \mathbf{A})\hat{\mathbf{r}}]$$

$$\mathbf{E}_{\text{inter}} = -c[\hat{\mathbf{r}}(\nabla \cdot (\mathbf{A}e^{-ikr})) + \nabla(\hat{\mathbf{r}} \cdot \mathbf{A}e^{-ikr})]e^{ikr}$$

$$\mathbf{E}_{\text{near}} = \frac{ic}{k}[\nabla(\nabla \cdot (\mathbf{A}e^{-ikr})) - \nabla^2(\mathbf{A}e^{-ikr})]e^{ikr}$$

Densities

densities of energy and momenta

$$w = \frac{\epsilon_0}{4}(\mathbf{E}\cdot\mathbf{E}^* + c^2\mathbf{B}\cdot\mathbf{B}^*) \qquad \mathbf{p} = \frac{\epsilon_0}{2}\Re[\mathbf{E}\times\mathbf{B}^*] \qquad \mathbf{j} = \mathbf{r}\times\mathbf{p}$$

Theorem

The density of any conserved quantity Q that propagates radially with constant speed is required to be homogeneous in r of degree -2.

Densities

radiated densities

$$w_{\text{far}} = \frac{\epsilon_0}{4}(\mathbf{E}_{\text{far}} \cdot \mathbf{E}^*_{\text{far}} + c^2\mathbf{B}_{\text{far}} \cdot \mathbf{B}^*_{\text{far}})$$

$$\mathbf{p}_{\text{far}} = \frac{\epsilon_0}{2}\Re[\mathbf{E}_{\text{far}} \times \mathbf{B}^*_{\text{far}}]$$

$$\mathbf{j}_{\text{far}} = \quad ?$$

Note

The pure far field does not contribute to the radiated angular momentum density.

$\mathbf{j}_{\text{far}} = \mathbf{r} \times \mathbf{p}_{\text{next}}$ with $\mathbf{p}_{\text{next}} = \frac{\epsilon_0}{2}\Re[\mathbf{E}_{\text{far}} \times \mathbf{B}^*_{\text{near}} + \mathbf{E}_{\text{inter}} \times \mathbf{B}^*_{\text{far}}]$

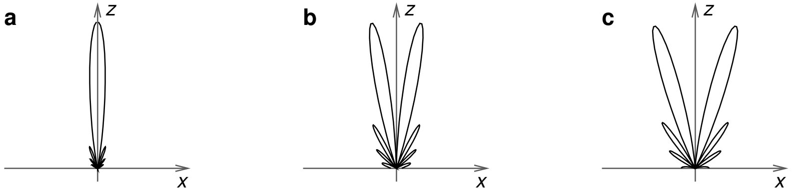
Results

$$w_{\mathsf{far}} = \frac{k^2}{4\mu_0}\left(\frac{I\mu_0 d}{8\pi r}\right)^2 (1+\cos^2\theta)(\mathcal{C}^*\mathcal{C})$$

$$\mathbf{p}_{\mathsf{far}} = \frac{\hat{\mathbf{r}}k^2}{4c\mu_0}\left(\frac{I\mu_0 d}{8\pi r}\right)^2 (1+\cos^2\theta)(\mathcal{C}^*\mathcal{C})$$

$$\mathbf{j}_{\mathsf{far}} = \frac{-\hat{\theta}k}{4c\mu_0}\left(\frac{I\mu_0 d}{8\pi r}\right)^2 \left\{ h\left(2\sin\theta(\mathcal{C}^*\mathcal{C}) - \cos\theta\frac{\partial}{\partial\theta}(\mathcal{C}^*\mathcal{C})\right) + l\,\frac{1+\cos^2\theta}{\sin\theta}(\mathcal{C}^*\mathcal{C})\right\}$$

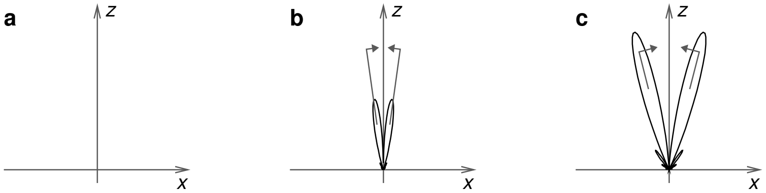
radiation patterns



Results

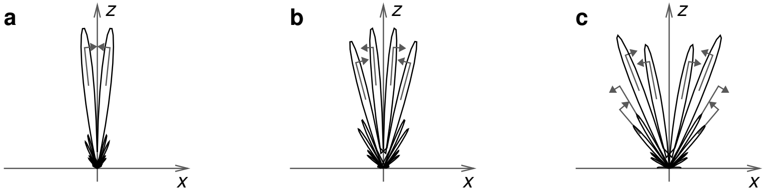
orbital angular momentum

$$\mathbf{l}_{\mathsf{far}} = l\frac{-\hat{\theta}k^2}{4\omega\mu_0}\left(\frac{I\mu_0 d}{8\pi r}\right)^2\left(\frac{1+\cos^2\theta}{\sin\theta}\right)(\mathcal{C}^*\mathcal{C})$$



spin angular momentum

$$\mathbf{s}_{\mathsf{far}} = h\frac{-\hat{\theta}k^2}{4\omega\mu_0}\left(\frac{I\mu_0 d}{8\pi r}\right)^2\left(2\sin\theta(\mathcal{C}^*\mathcal{C}) - \cos\theta\frac{\partial}{\partial\theta}(\mathcal{C}^*\mathcal{C})\right).$$



Fluxes

power, force, and torques

$$\frac{dW}{dt} = \int_S w_{\mathsf{far}} c r^2 d\Omega$$
$$\frac{d\mathbf{P}}{dt} = \int_S \mathbf{p}_{\mathsf{far}} c r^2 d\Omega$$
$$\frac{d\mathbf{L}}{dt} = \int_S \mathbf{l}_{\mathsf{far}} c r^2 d\Omega$$
$$\frac{d\mathbf{S}}{dt} = \int_S \mathbf{s}_{\mathsf{far}} c r^2 d\Omega$$

momenta

$\mathbf{P} = 0 \quad \rightarrow \quad \mathbf{J} = \mathbf{L} + \mathbf{S}$ intrinsic, but non-local

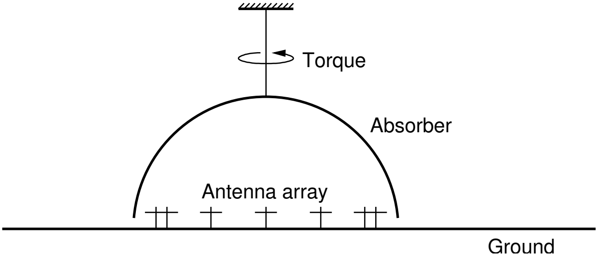
$L_z = \frac{l}{\omega} W \qquad l =$ vorticity

$S_z = \frac{h}{\omega} W \qquad h =$ helicity

Detection

torque measurement

$$\frac{dJ_z}{dt} = \alpha \frac{l+h}{\omega} \frac{dW}{dt} \qquad 0 < \alpha < 1$$



Spin and orbital angular momentum of a point dipole

vector potential

$$\mathbf{A}(\mathbf{r},t) = -\frac{\mu_0 d}{8\pi}\mathbf{I}(t)\frac{e^{ikr}}{r} \qquad \mathbf{I}(t) = -i\frac{2\omega}{d}\vec{\mu}(t) \qquad \vec{\mu}(t) = \begin{pmatrix}\mu_x\\ \mu_y\\ 0\end{pmatrix} e^{-i\omega t}$$

positive and negative helicity $\quad \vec{\mu} = \vec{\mu}^{+} + \vec{\mu}^{-}$

$$\vec{\mu}^{+} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\\ 0\end{pmatrix}\frac{\mu_x - i\mu_y}{\sqrt{2}}$$

$$\vec{\mu}^{-} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\\ 0\end{pmatrix}\frac{\mu_x + i\mu_y}{\sqrt{2}}$$

result

$$\frac{J_z}{W} = \frac{S_z}{W} = \frac{1}{\omega}\frac{|\vec{\mu}^{+}|^2}{|\vec{\mu}|^2} - \frac{1}{\omega}\frac{|\vec{\mu}^{-}|^2}{|\vec{\mu}|^2} = \frac{1}{\omega}i(\mu_x\mu_y^* - \mu_y\mu_x^*)/|\vec{\mu}|^2$$

Radio waves associated with longitudinal currents

currents $\quad n = 1 \ldots N$

$$\mathbf{I}_n = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{2\pi i l \frac{n}{N}}$$

vector potential

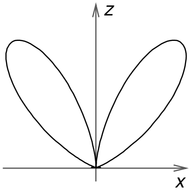
$$\mathbf{A} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mathcal{A}(r)\mathcal{C}(\theta, \phi)$$

densities of energy and momenta

$$w_{\mathsf{far}} = \frac{k^2}{4\mu_0}\left(\frac{I\mu_0 d}{8\pi r}\right)^2 (1 - \cos^2\theta)(\mathcal{C}^*\mathcal{C})$$

$$\mathbf{p}_{\mathsf{far}} = \frac{\hat{\mathbf{r}}k^2}{4c\mu_0}\left(\frac{I\mu_0 d}{8\pi r}\right)^2 (1 - \cos^2\theta)(\mathcal{C}^*\mathcal{C})$$

$$\mathbf{j}_{\mathsf{far}} = l\frac{-\hat{\theta}k^2}{4\omega\mu_0}\left(\frac{I\mu_0 d}{8\pi r}\right)^2 \left(\frac{1 - \cos^2\theta}{\sin\theta}\right)(\mathcal{C}^*\mathcal{C})$$



Thank you for your attention!

Växjö, June 17, 2008