ON ESTIMATION OF THE STOKES POLARIZATION PARAMETERS USING A PROBING TRIPOLE ARRAY

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ABSTRACT

In the paper we consider the estimation of the polarization properties of partially polarized electromagnetic waves using an array of Tripole antennas. We introduce the idea of a probing Tripole array consisting of three independent Robust Capon Beamformers, one for each vector component of the electromagnetic field, to cancel interfering signals coming from other directions than that under consideration. We develope three different estimators of the Stokes parameters used to classify the polarization properties of the electromagnetic wave; a Least-Square estimator, a Maximum Likelihood estimator and an estimator based on the rotation of the sample covariance matrix.

1. INTRODUCTION

During the last decade Radio-based Astronomy has started to exploit lower frequencies, $10-240$ MHz, in the study of Space. Two large astronomical arrays are under construction in nothern Europe, LOFAR[1] and LOIS[2]. In the LOIS project the Tripole antenna has been suggested as a possible antenna due to e.g. the Tripoles polarization flexibility [3]. One application within the LOIS project is to estimate the polarization properties of incoming electromagnetic(EM) waves to determine the properties of their origins. The objective of the probing Tripole array is twofolded; the array should separate the Signal of Interest(SOI), impinging from one particular direction-of-arrival(DOA), from EM waves impinging from other DOA:s using beamforming and thereafter estimate the polarization properties of the SOI. This procedure is repeated over a predefined DOA-space containing all possible DOA:s of interest. A different approach using Maximum Likelihood estimation of the DOA:s and polarization properties is made in [4] using a twin-dipole array and no beamforming. We believe that the concept of a probing Tripole array could perform better in this type of application due to weak SOI:s and the cancellation of strong man-made interferers. For Radio-based Astronomi the DOA-space can be limited to a hemisphere or even smaller if there exist a-priori information about the SOI. In section 2 we develope the signal model of the Tripole array receiving partially polarized EM waves and in section 3 we use Robust Capon Beamforming(RCB) [5] to separate the SOI from the interferers and mitigate the effects of steering vector errors which are most likely to occur in a large astronomical array. In section 4 we develope three estimators for the Stokes parameters, a Least-Square estimator, a Maximum Likelihood estimator and an estimator based on the rotation of the sample covariance matrix. In section 5 we exemplify the concept of the probing Tripole array using the three estimators by a simulation.

2. SIGNAL MODEL

Consider a Uniform Linear Array(ULA) aligned along the Cartesian base-vector $\hat{\boldsymbol{x}}$ constructed of M Tripoles. The placement vector for Tripole m is $\boldsymbol{r}_m = (m-1)d\hat{\boldsymbol{x}}$ where d is the distance between two adjecent Tripoles. We assume L incoming partially polarized EM waves where each EM wave has a distinct DOA and a arbitrary polarization state, see e.g. [6]. The cumulative EM field at position $\boldsymbol{r}$ is described by

$$\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{E}_0(\boldsymbol{r},t) + \sum_{l=1}^{L-1} \boldsymbol{E}_l(\boldsymbol{r},t), \qquad (1)$$

where we have separated $\boldsymbol{E}_0(\boldsymbol{r},t)$ or the SOI to emphashize that $\boldsymbol{E}_0(\boldsymbol{r},t)$ impings from the DOA(θ_0,ϕ_0) that shall be probed next and that $\{\boldsymbol{E}_l(\boldsymbol{r},t)\}_{l=1}^{L-1}$ are EM waves with different DOA:s than the SOI and we interpret them as interferers to be cancelled when probing (θ_0,ϕ_0). We assume that the incoming EM waves are plane-waves,

$$\boldsymbol{E}_l(\boldsymbol{r},t) = (s_{\phi_l}(t)\hat{\boldsymbol{\phi}}_l + s_{\theta_l}(t)\hat{\boldsymbol{\theta}}_l)e^{j\omega_c t - j\boldsymbol{k}_l\cdot\boldsymbol{r}}, \qquad (2)$$

l=0…L-1, where $\hat{\boldsymbol{\phi}}_l$ and $\hat{\boldsymbol{\theta}}_l$ are the base-vectors in spherical coordiantes and depends on the DOA (θ_l,ϕ_l) of the EM wave. Further, ω_c is the center angular frequency of operation, $\lambda = \omega_c/c$ is the wavelength, c is the propagation velocity, $\boldsymbol{k}_l = -2\pi/\lambda\hat{\boldsymbol{r}}_l$ is the wave-vector where $\hat{\boldsymbol{r}}_l$ is the spherical coordinate base-vector pointing in the DOA (θ_l,ϕ_l) towards the wavefront of the l:th wave with the position of the first Tripole as reference. Observe that $-\hat{\boldsymbol{r}}_l, \hat{\boldsymbol{\phi}}_l, \hat{\boldsymbol{\theta}}_l$ form a right-handed coordinate system for the

l:th EM wave while the dipole-elements of one Tripole are aligned along the Cartesian base-vectors $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}$. The signal components of one EM wave from (2) are (omitting the index l) $s_\phi(t)$ and $s_\theta(t)$ where both are assumed to be narrowband ergodic zero-mean complex Gaussian random processes with statistical properties described by the coherency matrix $\mathbf{J}$,[4] [6],

$$\mathbf{J} = \mathcal{E}\left\{\mathbf{s}(t)\mathbf{s}(t)^H\right\} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \quad (3)$$

where $\mathbf{s}(t) = [s_\phi(t)\ s_\theta(t)]^T$, $\mathcal{E}$ is the expectation operator and H and T is the Hermitian transponate and transponate, respectively. We now introduce the Stokes parameters, [6],

$$\begin{aligned} s_0 &= J_{11} + J_{22}, & (4)\\ s_1 &= J_{11} - J_{22}, & (5)\\ s_2 &= 2\Re\{J_{12}\}, \text{ and} & (6)\\ s_3 &= 2\Im\{J_{12}\}. & (7) \end{aligned}$$

The Stokes parameters are often used to calculate the power $\sigma^2 = \text{Tr}\{\mathbf{J}\} = s_0$, where Tr is the trace operator, and to describe the polarization state and degree of polarization(DoP) of an EM wave. The DoP P can be calculated as

$$P = \frac{\sqrt{s_1^2 + s_2^2 + s_3^2}}{s_0}, \quad (8)$$

where $0 \leq P \leq 1$. If the EM wave is completely polarized i.e. the EM wave has full degree of polarization, $P = 1$ and

$$\begin{aligned} s_1 &= s_0\cos(2\alpha)\cos(2\beta), & (9)\\ s_2 &= s_0\cos(2\alpha)\sin(2\beta), & (10)\\ s_3 &= s_0\sin(2\alpha), & (11) \end{aligned}$$

where α and β are the angles of the polarization ellipse that desribes the ellipticy and the orientation of the polarization ellipse, respectively. Here, $-\pi/4 \leq \alpha \leq \pi/4$ and $0 \leq \beta \leq \pi$, $\alpha > 0$ corresponds to clockwise polarization, $\alpha = \pm\pi/4$ corresponds to circular polarization and $\alpha = 0$ correspsonds to linear polarization. We can express $\mathbf{J}$ as a linear combination of the Stokes parameters,

$$\mathbf{J} = s_0\mathbf{F}_0 + s_1\mathbf{F}_1 + s_2\mathbf{F}_2 + s_3\mathbf{F}_3 \quad (12)$$

where

$$\mathbf{F}_0 = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}, \quad \mathbf{F}_1 = \begin{pmatrix} 0.5 & 0 \\ 0 & -0.5 \end{pmatrix},$$
$$\mathbf{F}_2 = \begin{pmatrix} 0 & 0.5 \\ 0.5 & 0 \end{pmatrix}, \quad \mathbf{F}_3 = j\begin{pmatrix} 0 & 0.5 \\ -0.5 & 0 \end{pmatrix}.$$

Now, assume that the dipole-elements of each Tripole are short compared to the wavelength of operation so that the output voltage of each dipole-element will be proportional to the strength of the components of EM vector-field. Since each dipole-element in a Tripole are aligned along a Cartesian base-vector, the output voltage from one Tripole will give a measurement of the complete EM field-vector expressed in Cartesian coordiantes. The separation of the components of the EM wave is an advantage when the objective is to characterize the incoming EM waves. We divide the Tripole array into three orthogonal dipole arrays, termed subarrays where each subarray measures one component of the EM vector-field in (2). After demodulation, filtering and sampling (we continue to use t as the variable for discrete time) we denote the output voltages of the subarrays as $\mathbf{y}_x(t), \mathbf{y}_y(t), \mathbf{y}_z(t)$ where the subscipts x, y, z refers to which Cartesian base-vector the dipole-elements of the subarray are aligned with. From (1) and (2),

$$\begin{aligned} \mathbf{y}_x(t) &= \mathbf{A}\mathbf{s}_x(t) + \mathbf{v}_x(t), & (13)\\ \mathbf{y}_y(t) &= \mathbf{A}\mathbf{s}_y(t) + \mathbf{v}_y(t), & (14)\\ \mathbf{y}_z(t) &= \mathbf{A}\mathbf{s}_z(t) + \mathbf{v}_z(t), & (15) \end{aligned}$$

where $\mathbf{A} = [\mathbf{a}_0 \ldots \mathbf{a}_{L-1}]$ is the array manifold matrix and $\mathbf{a}_l = \left[\left\{e^{-j\boldsymbol{k}_l \cdot \boldsymbol{r}_m}\right\}_{m=1}^{M}\right]$ is the steering vector for the l:th signal. The elements of the signal column matrices are

$$\begin{aligned} [\mathbf{s}_x(t)]_l &= -\sin\phi_l s_{\phi_l}(t) + \cos\theta_l\cos\phi_l s_{\theta_l}(t), & (16)\\ [\mathbf{s}_y(t)]_l &= \cos\phi_l s_{\phi_l}(t) + \cos\theta_l\sin\phi_l s_{\theta_l}(t), & (17)\\ [\mathbf{s}_z(t)]_l &= -\sin\theta_l s_{\theta_l}(t), & (18) \end{aligned}$$

and the column vectors $\mathbf{v}_x(t), \mathbf{v}_y(t), \mathbf{v}_z(t)$ are independent zero-mean white Gaussian noise vectors with equal variance σ_v^2. In the next section we will use the three subarray models to develop a probing-array using beamforming

3. PROBING AND BEAMFORMING

From section 2 we have three independent subarrays (13)-(15), each subarray measuring one component of the cumulative incident EM vector-field in (1). We intend to estimate the polarization properties of an EM wave incident on a distinct DOA while all other interferers are cancelled. The DOA of interest could have been previously estimated or it is treated as the next DOA in a predefined DOA-space that shall be probed. Now, we define the probing Tripole array as three independent Robust Capon Beamformers where each Robust Capon Beamfomer(RCB) operats on one subarray of dipoles to separate the SOI from the interferers. RCB was recently proposed as a roboust beamforming technique when there is limited knowledge about the true steering vectors, see [5] and references therein. This is most likely the case in a practical situation where there might be array calibration errors e.g. from mutual coupling between the subarrays or DOA estimation errors. Define the spatially filtered signals as

$$\mathbf{x}(t) = \begin{pmatrix} \hat{\mathbf{w}}_x^H \mathbf{y}_x(t) \\ \hat{\mathbf{w}}_y^H \mathbf{y}_y(t) \\ \hat{\mathbf{w}}_z^H \mathbf{y}_z(t) \end{pmatrix}, \tag{19}$$

where $\hat{\mathbf{w}}_x, \hat{\mathbf{w}}_y, \hat{\mathbf{w}}_z$ are the optimal weights per subarray obtained using RCB. Assuming that the degrees of freedom M of the beamforming is large enough so that the beamformers are capable of suppressing all interferers we can rewrite (19) using (13)-(18) as

$$\mathbf{x}(t) = \mathbf{W}\mathbf{V}\mathbf{s}(t) + \mathbf{v}(t) \tag{20}$$

where $\mathbf{W} = \text{diag}\left[\hat{\mathbf{w}}_x^H \mathbf{a}_0 \; \hat{\mathbf{w}}_y^H \mathbf{a}_0 \; \hat{\mathbf{w}}_z^H \mathbf{a}_0\right]$, $\mathbf{s}(t) = [s_{\phi_0}(t) \; s_{\theta_0}(t)]^T$ and

$$\mathbf{V} = \begin{pmatrix} -\sin(\phi_0) & \cos(\theta_0)\cos(\phi_0) \\ \cos(\phi_0) & \cos(\theta_0)\sin(\phi_0) \\ 0 & -\sin(\theta_0) \end{pmatrix} \tag{21}$$

is a rotation matrix with orthogonal columns. Further, $\mathbf{v}(t) = \left[\hat{\mathbf{w}}_x^H \mathbf{v}_x(t) \; \hat{\mathbf{w}}_y^H \mathbf{v}_y(t) \; \hat{\mathbf{w}}_z^H \mathbf{v}_z(t)\right]^T$ is a zero-mean Gaussian noise vector with a covariance matrix

$$\mathbf{R}_v = \sigma_v^2 \text{diag}\left[\hat{\mathbf{w}}_x^H \hat{\mathbf{w}}_x \; \hat{\mathbf{w}}_y^H \hat{\mathbf{w}}_y \; \hat{\mathbf{w}}_z^H \hat{\mathbf{w}}_z\right] = \sigma_v^2 \mathbf{G}. \tag{22}$$

Observe that Standard Capon Beamforming(SCB)[7] incorporates the linear constraint $\mathbf{w}^H \mathbf{a}_0 = 1$ resulting in $\mathbf{W} = \mathbf{I}$ where $\mathbf{I}$ is a unity matrix. This is not the case for RCB. RCB introduce $\mathbf{W}$ that has a diagonal structure and can in this case be interpreted as complex gains, one gain for each subarray. We do not possess information about the elements of $\mathbf{W}$ since we do not know $\mathbf{a}_0$. We also observe that $\mathbf{W}$ is dependent on the DOA of the SOI. The covariance matrix for $\mathbf{x}(t)$ is

$$\mathbf{C} = \mathcal{E}\left\{\mathbf{x}(t)\mathbf{x}(t)^H\right\} = \mathbf{W}\mathbf{V}\mathbf{J}\mathbf{V}^H\mathbf{W}^H + \sigma_v^2\mathbf{G} \tag{23}$$

where $\mathbf{J}$ is the coherency matrix (3) of the SOI. In the next section we use (23) to estimate the Stokes parameters by a Least-Square method, a Maximum Likelihood method and a Rotation method.

4. STOKES PARAMETER ESTIMATION

We consider the estimation of the Stokes parameters $\mathbf{s}_{st} = [s_0 \; s_1 \; s_2 \; s_3]^T$ and the noise variance σ_v^2 by a Least-Square method and a Maximum Likelihood Method. The two methods are compared to the Rotation method which is a common method in Radio-based Astronomy. In these three methods we assume that the model developed in section 3 is valid i.e. the beamformer of each subarray has cancelled all interferers and the SOI is described by (20) and (23). We are aware of the $\mathbf{W}$ matrix introduced by the RCB of which we have no information and therefore make the assumption that $\mathbf{W} = \mathbf{I}$ in (23). The errors introduced due to this assumption are incorporated in the steering vector errors.

4.1. Least-Square Estimator

Using (12) and (23) with $\mathbf{W} = \mathbf{I}$ we find the Stokes parameters for the SOI as

$$[\hat{\mathbf{s}}_{st}^T \; \hat{\sigma}_v^2]^T = \arg \min_{s_0, s_1, s_2, s_3, \sigma_v^2} \left\| \hat{\mathbf{C}} - \mathbf{C}_{LS} \right\|_F^2 \tag{24}$$

where $\hat{\mathbf{C}} = 1/N \sum_{t=1}^{N} \mathbf{x}(t)\mathbf{x}(t)^H$ is the sample covariance matrix, N is the number of snapshots, T is the matrix transponate, F denotes the Frobenius norm and

$$\mathbf{C}_{LS} = \sum_{k=0}^{3} s_k \mathbf{V}\mathbf{F}_k\mathbf{V}^H + \sigma_v^2\mathbf{G}. \tag{25}$$

4.2. Maximum Likelihood Estimator

We consider the stochastic maximum likelihood estimator for $\mathbf{J}$ in (23) with $\mathbf{W} = \mathbf{I}$. For the case of white Gaussian noise see e.g. [8] for $\hat{\mathbf{J}}$ (denoted $\hat{\mathbf{R}}_s$ in [8]) which we modify to account for the colored noise described by (22). The ML estimators for $\mathbf{J}$ and σ_v^2 are

$$\hat{\mathbf{J}}_{ML} = (\mathbf{V}^+(\hat{\mathbf{C}} - \hat{\sigma}_v^2\mathbf{G})\mathbf{V})^{+H} \tag{26}$$

$$\mathbf{V}^+ = (\mathbf{V}^H\mathbf{G}^{-1}\mathbf{V})^{-1}\mathbf{V}^H\mathbf{G}^{-1} \tag{27}$$

and

$$\hat{\sigma}_{vML}^2 = \text{Tr}\left\{\mathbf{G}^{-1}(\mathbf{I} - \mathbf{V}\mathbf{V}^+)\hat{\mathbf{C}}\right\}. \tag{28}$$

Using (26)-(28) in (4)-(7) we obtain the estimated Stokes parameters.

4.3. Rotation of the Sample Covariance Matrix

First, we estimate σ_v^2 as the minimum eigenvalue of the matrix pencil $(\hat{\mathbf{C}}, \mathbf{G})$ i.e. find the eigenvectors $\{\mathbf{e}_k\}_{k=1}^3$ and corresponding eigenvalues $\{\tilde{\lambda}_k\}_{k=1}^3$ that fullfills

$$\hat{\mathbf{C}}\mathbf{e}_k = \tilde{\lambda}_k \mathbf{G}\tilde{\mathbf{e}}_k, \; k = 1, 2, 3, \tag{29}$$

and let

$$\hat{\sigma}_{vR}^2 = \min\{\tilde{\lambda}_k\}. \tag{30}$$

Observing that $\mathbf{V}$ has orthogonal columns and $\mathbf{V}^H\mathbf{V} = \mathbf{I}$ we define

$$\hat{\mathbf{J}}_R = \mathbf{V}^H(\hat{\mathbf{C}} - \hat{\sigma}_v^2\mathbf{G})\mathbf{V} \tag{31}$$

which is easily computed. Then, use (31) in (4)-(7) to get the estimated Stokes parameters.

5. SIMULATIONS

We display a simulation using the three methods in section 4 and the signal model developed in sections 2 and 3 to estimate the Stoke parameters of the SOI. A complete search over a DOA-space is not included but the concept has been illustrated. Details on how the weight vectors in (19) are obtained using RCB can be found in [5]. For simplicity we consider a one dimensional DOA $(\theta = 45°, \phi)$. For each subarray, the error in the steering vector $\mathbf{a}_0$ is modeled as a DOA error $\bar{\mathbf{a}} = \mathbf{a}_0(\phi_0 + \Delta)$, $\Delta = 3°$, where $\bar{\mathbf{a}}$ is a measurement of $\mathbf{a}_0$ at hand and together with $\epsilon = 3$ are the user parameters of the RCB algorithm, see [5] for further details. We let each subarray beamformer use the theoretical covariance matrix $\mathbf{C}_k = \mathcal{E}\left\{\mathbf{y}(t)_k \mathbf{y}_k^H(t)\right\}$, $k = x, y, z$ obtained using (13)-(15). Using the corresponding sample covariance matrices affects the performance of the beamformers but in [5] it is shown that RCB works well even with a few snapshots. Also, we use an array of $M = 10$ Tripoles, $d = \lambda/2$ and we assume $L = 5$ signals. The power of the SOI $\sigma_0^2 = 10$, the power of the interferers $\sigma_l^2 = 100, l = 1...4$, and $\sigma_v^2 = 10$. Further, we let the DOA of the SOI $\phi_0 = 45°$ and the DOA:s of the interferers $\phi = \{20°\ 67°\ 95°\ 130°\}$. The polarization state of the SOI is given by the corresponding Stokes parameters $s_0 = 10$, $s_1 = -6.66$, $s_2 = 5.21$, $s_3 = 1.49$ while the interferers are given arbitrary polarization states. The specific polarization states of the interferers are of less importance since there can exist combinations of DOA:s and polarization states that will have a negative affect on the beamforming but here we have assumed that the beamformers will cancel all interferers. We use 1000 Monte-Carlo simulations to obtain a mean-value estimate of the Stokes parameters for different number of snapshots N. The results are displayed in figure 1-4. First, we observe that the three

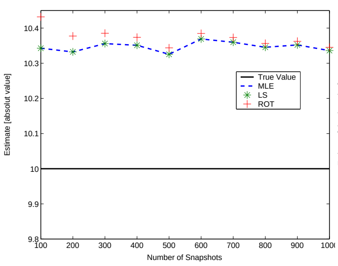

Fig. 1. Estimation of s_0

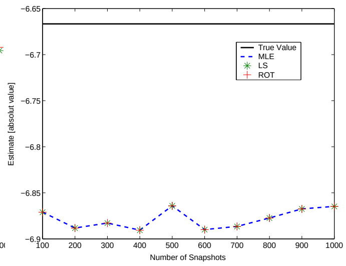

Fig. 2. Estimation of s_1

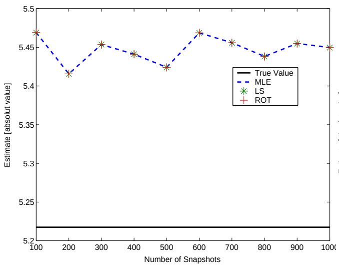

Fig. 3. Estimation of s_2

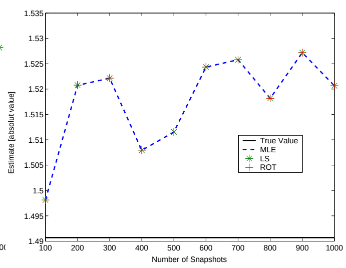

Fig. 4. Estimation of s_3

methods have a bias related to not knowing $\mathbf{W}$ in (23). Using SCB and assuming no steering vector error, simulations shows that the estimates of all three methods approches the correct values as N increases. The estimation could most likely be improved by estimation of $\mathbf{W}$ but it is not considered here. More over, it is noted that the LS-method and ML-method have a simular performance for this simulation while the Rotation method has a larger bias in the estimate of s_0. If the noise estimation was removed from the Rotation method larger bias would be expected. We also point out that the good results obtained in this simulation are due to many parameters but most important are the performance of beamformers. Using more antennas and if the uncertainty about the steering vectors remains there would be a larger bias and if possible compensation should be applied e.g. estimation of $\mathbf{W}$ or further array calibration.

6. SUMMARY

We have introduced a probing Tripole array in the context of polarization properties estimation of partially polarized EM waves. The concept exploites the Tripoles polarization flexibility to measure the components of the EM vectorfield combined with the robustness of the RCB technique to mitigate array steering vector uncertainties when eliminating interferers and separating signals from different DOA:s. Three different estimators of the Stokes parameters are introduced and exemplified in a simulation.

7. REFERENCES

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